JOURNAL OF APPROXIMATION THEORY 13, 167–177 (1975)

Approximation by Families of Multipliers for (C, α)-Bounded Fourier Expansions in Locally Convex Spaces. I. Order-Preserving Operators

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Communicated by P. L. Butzer

DEDICATED TO PROFESSOR G. G. LORENTZ ON THE OCCASION OF HIS SIXTY-FIFTH BIRTHDAY

1. INTRODUCTION

In some previous papers [1, 4, 11] it was shown that certain fundamental problems concerning the approximation by linear operators may be treated from a unified point of view in arbitrary Banach spaces. The operators in question were assumed to be generated via summation processes of abstract Fourier expansions. The principal tool was a multiplier criterion based upon Cesàro (or Riesz) summability.

In this note we would like to make some initial remarks concerning extension to locally convex spaces. To this end, we first describe, in Section 2, the general procedure. For the most familiar orthogonal expansions and spaces the partial sums are already equicontinuous so that further summability conditions and multiplier criteria may be dispensed with. The situation changes immediately, however, if one restricts the discussion to order-preserving operators (cf. (2.7)) in which one may be interested in connection with certain problems in approximation theory (cf. (6.2)). Therefore we proceed with these operators in countably normed spaces, the corresponding criterion being given by Theorem 1. To study some specific examples of expansions and spaces, Section 3 treats Hermite series in certain weighted function spaces, Section 4 deals with some spaces of (smooth) test functions, and Section 5 is concerned with their duals, thus with countable union spaces. As an example of the approximation-theoretic problems which may be considered, Section 6 compares two different processes with respect to their rate of convergence.

* The contribution of this author was supported by DFG-Grant Ne 171/1.

640/13/2-5

2. General Theory

Let **Z**, **P**, **N** denote, respectively, the set of all integers, all nonnegative integers, and all positive ones. Let X be a CNS, i.e., a complete countably normed linear space, the topology being generated by a monotone increasing sequence of norms $\{a_k\}_{k\in\mathbf{P}}$ which are assumed to be in concordance (cf. [2, p. 5 ff.; 3, p. 10 ff.]). Thus, if X_k denotes the completion of X by a_k, then $X_0 \supset X_1 \supset \cdots \supset X$ and $X = \bigcap_{k=0}^{\infty} X_k$. Let $[X]$ be the class of all continuous linear operators of X into itself and $\{P_k\}_{k\in\mathbf{P}} \subset [X]$ be a total sequence of mutually orthogonal projections. Then with each $f \in X$ one may associate its unique Fourier series expansion

$$f \sim \sum_{k=0}^{\infty} P_k f, \qquad (f \in X). \tag{2.1}$$

With s the set of all sequences $\tau = \{\tau_k\}_{k\in\mathbf{P}}$ of scalars, $\tau \in s$ is called (cf. [5]) a multiplier for X (with respect to $\{P_k\}$) if for each $f \in X$ there exists an element $f^\tau \in X$ such that

$$P_k f^\tau = \tau_k P_k f, \qquad (k \in \mathbf{P}). \tag{2.2}$$

Since $\{P_k\}$ is total, f^τ is uniquely determined by f. The corresponding multiplier operator T, given via $Tf := f^\tau$, is a closed linear operator from X into itself which is continuous by the closed graph theorem (cf. [9, p. 126]). The set of all multipliers τ for X is denoted by M, the corresponding set of multiplier operators T by $[X]_M$.

To derive a multiplier criterion one may proceed as in [1, 4, 11]. Thus, let the (C, α)-means of (2.1) be defined by

$$(C, \alpha)_n f := (A_n^\alpha)^{-1} \sum_{k=0}^{n} A_{n-k}^\alpha P_k f, \qquad A_n^\alpha := \binom{n+\alpha}{n}, \tag{2.3}$$

and assume that, for some $\alpha \geq 0$, the operators $(C, \alpha)_n$ are equicontinuous, i.e., for each $k \in \mathbf{P}$ there exists some $l \in \mathbf{P}$ and a constant $C_\alpha(k, l)$ such that

$$a_k((C, \alpha)_n f) \leq C_\alpha(k, l)\, a_l(f), \qquad (f \in X), \tag{2.4}$$

$C_\alpha(k, l)$ being independent of f and n. Let $l^\infty \subset s$ be the set of bounded sequences and

$$bv_{\alpha+1} := \left\{\tau \in l^\infty;\ \|\tau\|_{bv_{\alpha+1}} := \sum_{k=0}^{\infty} A_k^\alpha\, |\Delta^{\alpha+1}\tau_k| + \lim_{k\to\infty} |\tau_k| < \infty\right\}, \tag{2.5}$$

the (fractional) difference operator being defined via (for the existence of $\tau_\infty := \lim_{k\to\infty} \tau_k$ and further details cf. [11] and the literature cited there)

$$\Delta^\alpha \tau_k := \sum_{m=0}^{\infty} A_m^{-\alpha-1} \tau_{k+m} = \sum_{m=0}^{\infty} (-1)^m \binom{\alpha}{m} \tau_{k+m} . \tag{2.6}$$

Then one has $bv_{\alpha+1} \subset M$ provided (2.4) holds (cf. proof of Theorem 1).

However, for the classical orthogonal expansions and spaces to be considered in the following, one usually has that the partial sums are already equicontinuous, i.e., (2.4) holds for $\alpha = 0$. This implies that each element $\tau \in bv_1$, the set of sequences of bounded variation, is a multiplier for X, and there is no need for a distinguished theory. This will become quite different if one restricts oneself to a suitable subclass of operators arising in connection with certain problems in approximation theory (cf. (6.2)).

To this end, $F \in [X]$ will be called order-preserving (for the motivation see Section 5) if for each $k \in \mathbf{P}$ there exists a constant B_k such that

$$a_k(Ff) \leqslant B_k a_k(f), \qquad (f \in X). \tag{2.7}$$

Thus F has a (unique) bounded linear extension to the whole Banach space X_k. In particular, a multiplier $\tau \in M$ is called order-preserving if the corresponding multiplier operator T satisfies (2.7). In this case we set

$$\| \tau \|_{M;k} := \sup\{a_k(Tf);\, a_k(f) \leqslant 1\} = \| T \|_{[X_k]} . \tag{2.8}$$

Correspondingly, we suppose that the (C, α)-means are uniformly order-preserving for some $\alpha \geqslant 0$, i.e., for each $k \in \mathbf{P}$ there exists a constant $C_\alpha(k)$ such that

$$a_k((C, \alpha)_n f) \leqslant C_\alpha(k)\, a_k(f), \qquad (f \in X), \tag{2.9}$$

$C_\alpha(k)$ being independent of f and n. Then

THEOREM 1. *Let $\{P_k\} \subset [X]$ be a total sequence of mutually orthogonal projections satisfying* (2.9) *for some $\alpha \geqslant 0$. Then every $\tau \in bv_{\alpha+1}$ is an order-preserving multiplier and*

$$\| \tau \|_{M;k} \leqslant C_\alpha(k) \, \| \tau \|_{bv_{\alpha+1}} . \tag{2.10}$$

Proof. Analogously to [1] we set up for each $f \in X$

$$f^\tau := \sum_{j=0}^{\infty} A_j^\alpha \Delta^{\alpha+1} \tau_j \cdot (C, \alpha)_j f + \tau_\infty f. \tag{2.11}$$

Then $f^\tau \in X_k$ for each $k \in \mathbf{P}$ since by (2.9)

$$a_k(f^\tau) \leq C_\alpha(k)\, a_k(f) \sum_{j=0}^{\infty} A_j^\alpha \,|\Delta^{\alpha+1}\tau_j| + |\tau_k|\, a_k(f)$$
$$\leq C_\alpha(k)\, \|\tau\|_{bv_{\alpha+1}}\, a_k(f).$$

Hence, $f^\tau \in X = \bigcap_{k=0}^{\infty} X_k$ is well-defined for each $f \in X$, and (2.10) follows. Moreover, $P_k f^\tau = \tau_k P_k f$ for each $k \in \mathbf{P}$ since (cf. [11, p. 20] and the literature cited there)

$$\tau_n = \sum_{k=0}^{\infty} A_k^\alpha \Delta^{\alpha+1}\tau_{k+n}\,. \tag{2.12}$$

For an application of Theorem 1 to an approximation-theoretical question we refer to Section 6.

3. Weighted Function Spaces

Let L^p, $1 \leq p \leq \infty$, denote usual Lebesgue spaces (with respect to ordinary Lebesgue measure), thus, with $\mathbf{R}$ the set of real numbers, $L^p(\mathbf{R})$ is the set of functions for which the norms

$$\|f\|_p := \left(\int_{-\infty}^{\infty} |f(x)|^p\, dx\right)^{1/p}, \quad 1 \leq p < \infty; \qquad \|f\|_\infty := \operatorname*{ess\,sup}_{x\in\mathbf{R}} |f(x)|$$

are finite, respectively. $L_{\mathrm{loc}}^p(\mathbf{R})$ denotes the set of functions which belong locally to L^p, i.e., on every compact subset of $\mathbf{R}$.

Let us now consider some results of Muckenhoupt [6] from the point of view of Section 2. For the weights

$$U_r(x) := (1 + |x|)^r \exp\{-x^2/2\}, \qquad (x, r \in \mathbf{R}) \tag{3.1}$$

let us introduce the Banach spaces

$$X^{p,r} := \{f \in L_{\mathrm{loc}}^p(\mathbf{R});\, a_r(f) := \|f(x)\, U_r(x)\|_p < \infty\}. \tag{3.2}$$

If the indices r vary over some open interval $E \subset \mathbf{R}$, then $X_E^p := \bigcap_{r\in E} X^{p,r}$ is a CNS. We would like to treat Hermite expansions on X_E^p. Thus, if $H_n(x)$ is the nth Hermite polynomial given via (cf. [10, p. 101f])

$$\sum_{n=0}^{\infty} (H_n(x)/n!)\, s^n = \exp\{2xs - s^2\},$$

to each $f \in X_E{}^p$ one may associate its (well-defined, cf. [6I, p. 423], [8, p. 17]) Hermite series expansion

$$f \sim \sum_{k=0}^{\infty} P_k f, \tag{3.3}$$

where

$$(P_k f)(x) := \left(\frac{1}{2^k k!\,(\pi)^{1/2}} \int_{-\infty}^{\infty} f(u)\, H_k(u) \exp\{-u^2\}\, du\right) H_k(x). \tag{3.4}$$

For this expansion together with norms a_r given via (3.2) it follows that condition (2.9) is satisfied for $\alpha = 0$ provided (cf. [6II])

$$p \in (\tfrac{4}{3}, 4), \qquad -1/p < r < 1 - (1/p), \tag{3.5}$$

and for $\alpha = 1$ provided (cf. [8])

$$\begin{aligned} 1 \leqslant p \leqslant \infty, \qquad & r < -(1/p) + 3, && \text{if } 1 \leqslant p \leqslant 4, \\ & r < (1/3p) + (\tfrac{8}{3}), && \text{if } 4 < p \leqslant \infty, \\ & r > (1/3p) - 3, && \text{if } 1 \leqslant p < \tfrac{4}{3}, \\ & r > -(1/p) - 2, && \text{if } \tfrac{4}{3} \leqslant p \leqslant \infty. \end{aligned} \tag{3.6}$$

Thus for each fixed p, one obtains an open interval $E(\alpha, p)$ of indices r such that the (C, α)-means of (3.3) are uniformly order-preserving on $X^r_{E(\alpha,p)}$ for $\alpha = 0$ and $\alpha = 1$ according to (3.5) and (3.6), respectively. For example, one has (precisely) $E(0, 2) = (-\tfrac{1}{2}, \tfrac{1}{2})$, but $E = (1, 2) = (-\tfrac{5}{2}, \tfrac{5}{2})$.

Correspondingly one may treat Laguerre expansions in suitable weighted function spaces using results of [6], [8] (see also [7] for Jacobi series in case $\alpha = 0$).

4. Spaces of Test Functions

Let us now fit results developed by Zemanian [12, Chapter IX] into our setting. Consider $L^p = L^p(I)$ for some open interval $I := (c, d)$, $-\infty \leqslant c < d \leqslant \infty$. For $f \in L^p$, $g \in L^{p'}$, $(1/p) + (1/p') = 1$, set

$$(f, g) := \int_I f(u)\, \overline{g(u)}\, du, \tag{4.1}$$

the bar denoting complex conjugates. For certain positive integers n_j let $\mathscr{R}$ be a linear differential operator of the form

$$\mathscr{R} := \theta_0(x)(d/dx)^{n_1}\, \theta_1(x)(d/dx)^{n_2} \cdots (d/dx)^{n_\nu}\, \theta_\nu(x), \tag{4.2}$$

with infinitely differentiable functions $\theta_j(x) \neq 0$ on I satisfying

$$\mathscr{R} = \overline{\theta_\nu}(x)(-d/dx)^{n_\nu} \cdots (-d/dx)^{n_2} \overline{\theta_1}(x)(-d/dx)^{n_1} \overline{\theta_0}(x). \tag{4.3}$$

Let $\{\lambda_k\}_{k\in\mathbf{P}}$ with $|\lambda_0| \leqslant |\lambda_1| \leqslant \cdots$, $|\lambda_k| \to \infty$, be a sequence of real eigenvalues of $\mathscr{R}$ corresponding to an orthonormal sequence $\{\psi_k(x)\} \subset \bigcap_{1\leqslant p\leqslant\infty} L^p(I)$ of infinitely differentiable eigenfunctions, thus

$$\mathscr{R}\psi_k = \lambda_k\psi_k, \qquad (\psi_k, \psi_n) = \delta_{k,n}. \tag{4.4}$$

Then the testing function space $\mathscr{A}^p$ is defined to be the set of all complex-valued, infinitely differentiable functions φ on I such that

$$a_k^p(\varphi) := \sup_{0\leqslant j\leqslant k} a_0^p(\mathscr{R}^j\varphi), \qquad a_0^p(\varphi) := \|\varphi\|_p, \tag{4.5}$$

is finite for each $k \in \mathbf{P}$ and

$$(\mathscr{R}^k\varphi, \psi_n) = (\varphi, \mathscr{R}^k\psi_n), \tag{4.6}$$

for each $k, n \in \mathbf{P}$ (so that $\mathscr{R}$ turns out to be self-adjoint on $\mathscr{A}^p$). It follows that $\mathscr{A}^p$ is a CNS. For example, one may choose $I = \mathbf{R}$, $\mathscr{R} = (d/dx)^2 - x^2 + 1$, $\lambda_k = -2k$, and (cf. (3.4))

$$\psi_k(x) = (2^k k!\,(\pi)^{1/2})^{-1/2} \exp\{-x^2/2\}\, H_k(x), \qquad (k \in \mathbf{P}). \tag{4.7}$$

Here $\mathscr{A}^p = \mathfrak{S}$, the Schwartz space of testing functions of rapid descent.

Let again $\varphi \in \mathscr{A}^p$. Since $\psi_k \in L^{p'}$, the Fourier coefficients (φ, ψ_k) are well-defined so that with each φ one may associate the expansion

$$\varphi \sim \sum_{k=0}^{\infty} P_k\varphi, \qquad P_k\varphi := (\varphi, \psi_k)\,\psi_k. \tag{4.8}$$

In order to examine (2.4) for $\alpha = 0$, let there exist $j_0 \in \mathbf{P}$ such that

$$\begin{aligned} &\text{(i)} && \|\psi_k\|_p = O(\omega_p(k)) \qquad (k \to \infty), \\ &\text{(ii)} && {\sum_{k=0}^{\infty}}' \omega_p(k)\, \omega_{p'}(k)\, |\lambda_k|^{-j_0} < \infty, \end{aligned} \tag{4.9}$$

the dash indicating that those (finitely many) indices for which $\lambda_k = 0$ be omitted. In view of (4.4) one has

$$\begin{aligned} \mathscr{R}^j(C, 0)_n\, \varphi &= \sum_{k=0}^{n} (\varphi, \psi_k)\, \mathscr{R}^j\psi_k \\ &= {\sum_{k=0}^{n}}' \lambda_k^{j-j_0}(\varphi, \mathscr{R}^{j_0}\psi_k)\, \psi_k = {\sum_{k=0}^{n}}' \lambda_k^{-j_0}(\mathscr{R}^{j+j_0}\varphi, \psi_k)\, \psi_k, \end{aligned}$$

and hence by Hölder's inequality

$$\begin{aligned} a_m{}^p((C,0)_n\,\varphi) &= \sup_{0\leqslant j\leqslant m} \|\mathscr{R}^j(C,0)_n\,\varphi\|_p \\ &\leqslant \sup_{0\leqslant j\leqslant m} \sum_{k=0}^{n}{}' |\lambda_k|^{-j_0} \|\mathscr{R}^{j+j_0}\varphi\|_p \|\psi_k\|_{p'} \|\psi_k\|_p \\ &\leqslant B \sum_{k=0}^{n}{}' (\omega_p(k)\,\omega_{p'}(k)\,|\lambda_k|^{-j_0}) \cdot \sup_{0\leqslant j\leqslant m} \|\mathscr{R}^{j+j_0}\varphi\|_p \\ &\leqslant B_1 a^p_{m+j_0}(\varphi). \end{aligned}$$

so that the (C, α)-means of (4.8) are equicontinuous (in the sense of (2.4)) for $\alpha = 0$. On the other hand, the corresponding condition (2.9) is generally only satisfied for certain larger values of α, depending upon p and $\{P_k\}$.

Let us consider the trigonometric system in some detail. Thus choose $I = (-\pi, \pi)$, $\mathscr{R} = -i(d/dx)$ and, with a different numbering system,

$$\lambda_k = k, \qquad \psi_k(x) = (2\pi)^{-1/2}\, e^{ikx}, \qquad (k \in \mathbf{Z}).$$

Then $\mathscr{A}^p$ may be identified with $\mathscr{D}_{2\pi}$, the set of infinitely differentiable, 2π-periodic functions. On $\mathscr{D}_{2\pi}$ one may define a total sequence of orthogonal projections $\{P_k\}_{k\in\mathbf{P}}$ by

$$(P_0\varphi)(x) := \varphi\hat{\ }(0), \quad (P_k\varphi)(x) := \varphi\hat{\ }(k)\, e^{ikx} + \varphi\hat{\ }(-k)\, e^{-ikx} \quad (k \in \mathbf{N}), \tag{4.10}$$

$\varphi\hat{\ }(j)$ being the jth complex Fourier coefficient

$$\varphi\hat{\ }(j) := (2\pi)^{-1/2} \int_{-\pi}^{\pi} \varphi(u)\, e^{-iju}\, du \qquad (j \in Z).$$

Then the corresponding partial sum operators $(C, 0)_n$ are equicontinuous since (4.9) is satisfied for $j_0 = 2$ (note that $\|\psi_k\|_p = 1$ for all $k \in \mathbf{P}$, $1 \leqslant p \leqslant \infty$). On the other hand, whereas they are not uniformly order-preserving for $p = 1$, $p = \infty$, the $(C, 1)$-means do possess this property for all $1 \leqslant p \leqslant \infty$ since by Fejér's theorem

$$\begin{aligned} a_k{}^p((C,1)_n\,\varphi) &= \sup_{0\leqslant j\leqslant k} \|(C,1)_n\,\mathscr{R}^j\varphi\|_p \\ &\leqslant \sup_{0\leqslant j\leqslant k} \|\mathscr{R}^j\varphi\|_p = a_k{}^p(\varphi). \end{aligned}$$

Analogously one may deal with all the other classical orthogonal expansions such as those into Jacobi polynomials or Laguerre functions. Concerning the validity of a condition of type (2.9) one may as above consult the results in the corresponding Banach space-setting (cf. [1, 4, 11] and the literature cited there).

5. Countable Union Spaces

Let $\{Y_m\}_{m\in\mathbf{P}}$ be a monotonely increasing sequence of Banach spaces equipped with the monotonely decreasing sequence $\{\pi_m\}$ of norms, i.e.,

$$Y_m \subset Y_{m+1}, \qquad \pi_{m+1}(f) \leqslant \pi_m(f) \qquad \text{for all} \quad f \in Y_m. \tag{5.1}$$

Let $Y := \bigcup_{m=0}^{\infty} Y_m$ be the corresponding countable union space, i.e., the inductive limit of the Y_m's. Then a linear operator L from Y into itself is continuous if and only if the restriction to each Y_m is continuous, i.e., to each $m \in \mathbf{P}$ there exists some $j = j(m) \in \mathbf{P}$ such that $\pi_j(Lf) \leqslant C_{j,m}\pi_m(f)$ for all $f \in Y_m$ (cf. [2, p. 20]). In case one may take $j = m$, i.e.,

$$\pi_m(Lf) \leqslant C_m \pi_m(f) \qquad (f \in Y_m), \tag{5.2}$$

L is called order-preserving, thus being a bounded linear operator from Y_m into itself for each $m \in \mathbf{P}$. Of course, the constant C_m depends on the choice of $f \in Y$ as far as f determines $m \in \mathbf{P}$ via $f \in Y_m$. Obviously, one may proceed as in Section 2 to study order-preserving multipliers on Y.

A significant example of a countable union space is given by the dual $X^* = \bigcup_{k=0}^{\infty} X_k^*$ of a CNS $X = \bigcap_{k=0}^{\infty} X_k$, X_k^* being the dual of the Banach space X_k. Since the smallest k such that $f \in X^*$ belongs to X_k^* is called the order of the functional f (cf. [2, p. 10; 3, p. 28]), this motivates the terminology for operators satisfying (5.2) (or (2.7)). Of course, if condition (2.9) is valid on X for some $\alpha > 0$, then one may derive the corresponding one on X^* by duality. Particularly, if X is one of the testing function spaces of Section 4, this yields applications to multiplier operators as defined on the corresponding spaces of distributions. However, in this note we will not be concerned with these details.

6. A Comparison Theorem

In this section we would like to give an example of the approximation-theoretic problems which may be considered in the setting of Section 2. Thus let X be a CNS. A family $\{T(\rho)\}_{\rho>0} \subset [X]$ is called an approximation process if for each $k \in \mathbf{P}$

$$\lim_{\rho\to\infty} a_k(T(\rho)f - f) = 0 \qquad (f \in X). \tag{6.1}$$

Let $\{S(\rho)\} \subset [X]$ be a further approximation process. Then (cf. [11] and the literature cited there) one is interested in direct estimates between the quantities $T(\rho)f-f$ and $S(\rho)f-f$, thus to establish, for instance, the existence of constants $A_k > 0$ such that for each $k \in \mathbf{P}$

$$a_k(T(\rho)f-f) \leqslant A_k a_k(S(\rho)f-f) \qquad (f \in X, \rho > 0). \tag{6.2}$$

In this event, the process $\{T(\rho)\}$ is said to be better than $\{S(\rho)\}$ (in the order-preserving sense).

THEOREM 2. *Let $\{S(\rho)\}$, $\{T(\rho)\} \subset [X]_M$ with associated multipliers $\{\sigma_k(\rho)\}$, $\{\tau_k(\rho)\}$ be such that $\{\delta_k(\rho)\}$, given via*

$$\tau_k(\rho) - 1 = \delta_k(\rho)(\sigma_k(\rho) - 1) \qquad (k \in \mathbf{P}, \rho > 0),$$

defines a family of uniformly order-preserving multipliers. Then the process $\{T(\rho)\}$ is better than $\{S(\rho)\}$.

Proof. If $U(\rho) \in [X]_M$ corresponds to $\delta(\rho) \in M$, then

$$P_k(T(\rho)f-f) = \delta_k(\rho)(\sigma_k(\rho) - 1) P_k f = P_k(U(\rho)(S(\rho)f-f)).$$

Since $\{P_k\}$ is total, this implies

$$T(\rho)f-f = U(\rho)(S(\rho)f-f) \qquad (f \in X, \rho > 0),$$

and therefore (cf. (2.8)) for each $k \in \mathbf{P}$

$$a_k(T(\rho)f-f) \leqslant \|\delta(\rho)\|_{M;k}\, a_k(S(\rho)f-f) \qquad (f \in X, \rho > 0),$$

which gives the assertion since $\{\delta(\rho)\}$ is uniformly order-preserving.

Let X, $\{P_k\}$ be as in Section 2 such that condition (2.9) is satisfied for some $\alpha \geqslant 0$. Then one may consider the Abel–Cartwright and the Riesz means of (2.1), thus for $\kappa > 0$, $\rho > 0$

$$W_\kappa(\rho)f = \sum_{k=0}^{\infty} w\left(\left(\frac{k}{\rho}\right)^\kappa\right) P_k f, \qquad R_{\kappa,\lambda}(\rho)f = \sum_{k=0}^{\infty} r_\lambda\left(\left(\frac{k}{\rho}\right)^\kappa\right) P_k f, \tag{6.3}$$

where for $\lambda > 0$ (for the convergence of the first series cf. [1, Part II; 11, p. 23])

$$w(t) = e^{-t}, \qquad r_\lambda(t) = \begin{cases} (1-t)^\lambda, & 0 \leqslant t \leqslant 1, \\ 0, & 1 < t, \end{cases}$$

respectively. For each $\kappa > 0$, $\lambda > \alpha$ one has (cf. [1, Part II, 11]) that

$$\{w((k/\rho)^\kappa)\}, \qquad \{r_\lambda((k/\rho)^\kappa)\} \subset bv_{\alpha+1}$$

uniformly for $\rho > 0$. Moreover, if one sets

$$d(t) = [w(t) - 1]/[r_\lambda(t) - 1],$$

then it follows for each $\kappa > 0$, $\lambda > \alpha$ that (cf. [11, p. 58])

$$\{d((k/\rho)^\kappa)\}, \qquad \{1/d((k/\rho)^\kappa)\} \subset bv_{\alpha+1},$$

uniformly for $\rho > 0$. Therefore by Theorems 1, 2 the Abel–Cartwright means are better than the Riesz means, and vice versa, i.e., the processes (6.3) are equivalent for $\lambda > \alpha$. For integral α this holds true for $\lambda = \alpha$ as well.

Now one may specify the spaces X and projections $\{P_k\}$. For example, if one considers Hermite expansions on weight spaces according to Section 3, then for $\lambda \geq 1$

$$\begin{aligned}\|[W_\kappa(\rho) f(x) - f(x)]\, U_r(x)\|_p &\leq A_r \|[R_{\kappa,\lambda}(\rho) f(x) - f(x)]\, U_r(x)\|_p \\ &\leq B_r \|[W_\kappa(\rho) f(x) - f(x)]\, U_r(x)\|_p,\end{aligned}$$

for each $r \in E(1, p)$, the constants A_r, B_r being independent of $f \in X^p_{E(1,p)}$ and $\rho > 0$. Correspondingly, in the setting of Section 4 one has for $(\lambda \geq \alpha)\ \lambda > \alpha$

$$\begin{aligned}\sup_{0\leq j\leq k} \| W_\kappa(\rho)\, \mathscr{R}^j\varphi - \mathscr{R}^j\varphi \|_p &\leq A_k \sup_{0\leq j\leq k} \| R_{\kappa,\lambda}(\rho)\, \mathscr{R}^j\varphi - \mathscr{R}^j\varphi \|_p \\ &\leq B_k \sup_{0\leq j\leq k} \| W_\kappa(\rho)\, \mathscr{R}^j\varphi - \mathscr{R}^j\varphi \|_p\end{aligned}$$

for each $k \in \mathbf{P}$ and for all $f \in \mathscr{A}^p$, $\rho > 0$, in case the corresponding expansion (4.8) satisfies condition (2.9) for some (integral) $\alpha \geq 0$. Note that $\mathscr{R}^j W_\kappa(\rho)\varphi = W_\kappa(\rho)\, \mathscr{R}^j\varphi$ by examining the Fourier coefficients.

Acknowledgement

The authors express their sincere gratitude to Professor P. L. Butzer who read the manuscript and gave many valuable suggestions.

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